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Phase transitions in CSOS models

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Abstract. The exact solution of the CSOS models by Pearce and Seaton is analysed. This solution appears to describe a first-order transition without droplet singularities. We show that the system with free boundaries has no such transition, whereas the system with fixed boundary conditions has a first-order transition at an anisotropy dependent locus, unlike the findings of Pearce and Seaton. We also explain the absence of droplet singularities. Finally we present a simple Coulomb gas calculation of some critical exponents, resulting in the same expressions as given by Pearce and Seaton.

1. Introduction

Recently Pearce and Seaton presented a family of L-states cyclic solid-on-solid models that satisfy the Yang–Baxter equation [1]. In [2] they calculate the order parameters and the free energy per site for these models (in the thermodynamic limit) and they examine the corresponding critical behaviour. The order parameters possess jump discontinuities as a function of a parameter w_0, of which the free energy is independent. These discontinuities do not form a barrier against analytic continuation, unlike the behaviour of other first-order transitions. A well known example is the first-order transition in the d-dimensional ($d \geqslant 2$) ferromagnetic Ising model at low temperatures: based on studies of the droplet model [3] it has long been suspected that the free energy and the magnetization of the Ising model have no analytical continuation through zero magnetic field. This has been shown rigorously some years ago [4]. The *absence* of such 'droplet singularities' in the CSOS models must be understood.

This paper consists of three parts: in the first part we define the models and give a short review of the calculations given in [2] in order to see where the discontinuities in the order parameters come from; in the second part we resolve the paradox about the first-order transition without the droplet singularities and in the third part we give an alternative derivation of the critical exponents calculated for these models [2, 5, 6], based on Coulomb gas techniques.

2. The CSOS model on a square lattice

2.1. *Definition of the model*

A configuration of the system is specified by giving the heights at all lattice points. In the L-states model the heights can take on the integer values $0, 1, \ldots, L-1$, where $L \geqslant 3$. A configuration is allowed when heights on neighbouring sites differ by ± 1 mod L,

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so, following Pasquier [7], the model can be labelled by the Dynkin diagram of the Kac-Moody algebra $A^{(1)}_{L-1}$ (see figure 1).

The model is of the interaction round a face (IRF) type, so the Boltzmann weight of a configuration equals the product of the face weights. Therefore the model is completely specified by the face weights of the $6L$ configurations† that satisfy the adjacency condition.

A sufficient condition for integrability is the Yang-Baxter equation (YBE) [8]. For IRF models it takes the form:

For all $a, \ldots, f$:

$$\sum_g W\begin{pmatrix} a & g \\ b & c \end{pmatrix} W'\begin{pmatrix} f & e \\ a & g \end{pmatrix} W''\begin{pmatrix} e & d \\ g & c \end{pmatrix} = \sum_g W''\begin{pmatrix} f & g \\ a & b \end{pmatrix} W'\begin{pmatrix} g & d \\ b & c \end{pmatrix} W\begin{pmatrix} f & e \\ g & d \end{pmatrix}. \tag{2.1}$$

This equation can be represented graphically as in figure 2.

Pearce and Seaton's solution of the YBE is given by (see figure 3)

$$W\begin{pmatrix} a-1 & a \\ a & a+1 \end{pmatrix} = W\begin{pmatrix} a+1 & a \\ a & a-1 \end{pmatrix} = \alpha_a = \frac{\vartheta_1(\lambda-u,p)}{\vartheta_1(\lambda,p)} \tag{2.2a}$$

$$W\begin{pmatrix} a & a-1 \\ a+1 & a \end{pmatrix} = W\begin{pmatrix} a & a+1 \\ a-1 & a \end{pmatrix} = \beta_a = \frac{\vartheta_1(u,p)}{\vartheta_1(\lambda,p)} \frac{[\vartheta_4(w_{a-1},p)\vartheta_4(w_{a+1},p)]^{1/2}}{\vartheta_4(w_a,p)} \tag{2.2b}$$

$$W\begin{pmatrix} a & a+1 \\ a+1 & a \end{pmatrix} = \gamma_a = \frac{\vartheta_4(w_a+u,p)}{\vartheta_4(w_a,p)} \tag{2.2c}$$

$$W\begin{pmatrix} a & a-1 \\ a-1 & a \end{pmatrix} = \delta_a = \frac{\vartheta_4(w_a-u,p)}{\vartheta_4(w_a,p)} \tag{2.2d}$$

all other weights being equal to zero.

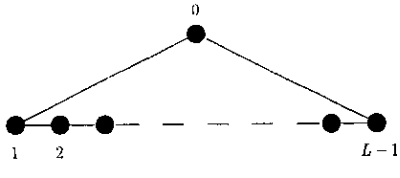

Figure 1. The adjacency graph for the L-states CSOS models.

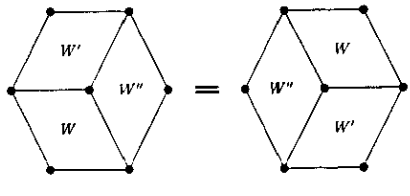

Figure 2. Graphical representation of the Yang-Baxter equation.

† When $L = 4$ there are 32 face configurations allowed by the adjacency condition. The 8 additional vortex-like configurations have zero Boltzmann weight in Pearce and Seaton's solution.

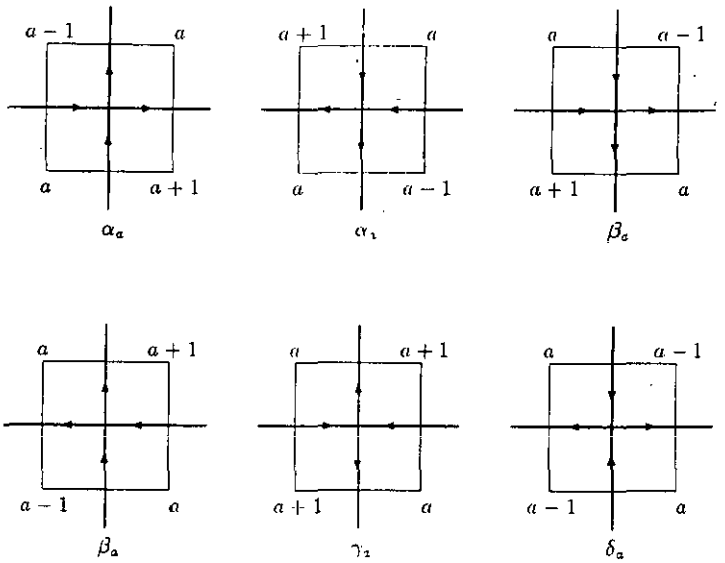

Figure 3. The allowed face configurations and weights for the CSOS models and the corresponding 6-vertex configurations.

Here

$$w_a = w_0 + a\lambda \tag{2.3}$$

$$\lambda = s\pi/L \qquad s \in \{1, \ldots, L-1\} \text{ and coprime to } L \tag{2.4}$$

and all heights are defined mod L.

The theta functions ϑ_1 and ϑ_4 are given by

$$\vartheta_1(u, p) = 2p^{1/4} \sin u \prod_{n=1}^{\infty} (1 - 2p^{2n} \cos 2u + p^{4n})(1 - p^{2n}) \tag{2.5a}$$

$$\vartheta_4(u, p) = \prod_{n=1}^{\infty} (1 - p^{2n-1} \cos 2u + p^{4n-2})(1 - p^{2n}). \tag{2.5b}$$

W' and W'' are given by the same expression in which u is replaced by u' and u'' respectively. These spectral parameters satisfy

$$u' = u + u''. \tag{2.6}$$

For a system with periodic boundary conditions the face weights $W\left(\begin{smallmatrix} d & c \\ a & b \end{smallmatrix}\right)$ may be given an extra factor $F(a, b)F(d, c)^{-1}G(a, d)G(b, c)^{-1}$ which cancels out of the configuration weight. The weight of a face with anisotropy parameter u is gauge equivalent to the weight of the $\pi/2$-rotated face with anisotropy $\lambda - u$:

$$W\left(\begin{matrix} d & c \\ a & b \end{matrix}\middle| u\right) = W\left(\begin{matrix} a & d \\ b & c \end{matrix}\middle| \lambda - u\right)\left[\frac{\vartheta_4(w_a, p)\vartheta_4(w_c, p)}{\vartheta_4(w_b, p)\vartheta_4(w_d, p)}\right]^{1/2}. \tag{2.7}$$

Because of the periodicity of the weights and gauge invariance the parameters u, w_0 and p can be chosen to satisfy

$$u \in (-\pi + \lambda, \lambda] \qquad w_0 \in [0, \pi) \qquad p \in (0, 1). \tag{2.8}$$

When $u \in (-\pi+\lambda, 0)$, however, the model is gauge equivalent to the model with

$$u' = -u \qquad s' = L - s \qquad p' = p \qquad w_0' = -w_0 \bmod \pi \tag{2.9}$$

so from now on we take

$$u \in [0, \lambda]. \tag{2.10}$$

2.2. *The critical limit*

In the limit $p \to 0$ the face weights become independent of a:

$$\alpha_a \to \frac{\sin(\lambda - u)}{\sin \lambda} = A \tag{2.11a}$$

$$\beta_a \to \frac{\sin u}{\sin \lambda} = B \tag{2.11b}$$

$$\gamma_a \to 1 = C \tag{2.11c}$$

$$\delta_a \to 1 = C. \tag{2.11d}$$

The model turns into a 6-vertex model with weights A, B and C (see figure 3). This 6-vertex model is critical [8, chapter 10] since

$$\Delta = \frac{A^2 + B^2 - C^2}{2AB} = -\cos \lambda \in (-1, 1). \tag{2.12}$$

When $u = \lambda/2$ and $p \to 0$, the model becomes the isotropic and critical F-model which can be described by a conformal field theory with central charge $c = 1$. Using its equivalence with a Coulomb gas and assuming certain scaling relations, the critical exponents can be readily determined, as shown in section 4.

2.3. *The ordered limit*

In studying the limit $p \to 1$ it is convenient to use a conjugate nome parametrization. Defining ε by

$$p = \exp(-\varepsilon) \tag{2.13}$$

and using the parameters

$$x = \exp\left(-\frac{\pi^2}{L\varepsilon}\right) \qquad w = \exp\left(-\frac{2\pi u}{\varepsilon}\right) \qquad v = \exp\left(-\frac{2\pi w_0}{\varepsilon}\right) \tag{2.14}$$

the face weights can be rewritten as

$$\alpha_a = \rho(u) w^{(L-s)/2L} \frac{E(x^{2s} w^{-1})}{E(x^{2s})} \tag{2.15a}$$

$$\beta_a = \rho(u) w^{(s-L)/2L} x^s \frac{E(w)[E(-vx^{2s(a+1)})E(-vx^{2s(a-1)})]^{1/2}}{E(x^{2s})E(-vx^{2sa})} \tag{2.15b}$$

$$\gamma_a = \rho(u) w^{(s-L)/2L + sa/L + \mu} \frac{E(-vx^{2sa} w)}{E(-vx^{2sa})} \tag{2.15c}$$

$$\delta_a = \rho(u) w^{(s+L)/2L - sa/L - \mu} \frac{E(-vx^{2sa} w^{-1})}{E(-vx^{2sa})} \tag{2.15d}$$

where $\rho(u) = \exp[u(\lambda - u)/\varepsilon]$, $\mu = w_0/\pi$ and $E(z) \equiv E(z, x^{2L})$, with

$$E(z, x) = \prod_{n=1}^{\infty} (1 - x^{n-1}z)(1 - x^n z^{-1})(1 - x^n). \tag{2.16}$$

The function E has some simple properties:

$$E(z, x) = E\left(\frac{x}{z}, x\right) = -zE\left(\frac{1}{z}, x\right) = -\frac{x}{z}E\left(\frac{z}{x}, x\right) \tag{2.17a}$$

and for $0 < a < b$:

$$\lim_{x \to 0} E(zx^a, x^b) = 1 \tag{2.17b}$$

$$\lim_{x \to 0} E(z, x^b) = 1 - z. \tag{2.17c}$$

The limit $p \to 1$ can be taken in many different ways. The most natural choice is to keep the original parameters u and μ fixed and let x approach zero. Another choice is to keep w and μ fixed and let x approach zero. In this limit the model becomes extremely anisotropic, since $u = O(\varepsilon) \to 0$; it is very convenient for calculating order parameters.

We define ground states† as those configurations that give the maximum contribution to the partition function *in the ordered limit.* Since we consider two different ordered limits we also have two types of ground states. In addition the ground states will depend on the boundary conditions. The ground states for a system with free boundary conditions are called 'bulk' ground states. These are calculated in [2] for the second (w fixed) ordered limit. When μ is not a multiple of $1/L$ they are the chequerboard configurations $(a, a+1)$ and $(a+1, a)$ with heights alternating between a and $a+1$ on the even and odd sublattices and with a satisfying

$$\left[\frac{sa}{L} + \mu\right] = \left[\frac{s(a+1)}{L} + \mu\right]. \tag{2.18}$$

Here $[x]$ denotes the integer part of x.

This equation has $L - s$ solutions, so there are $2(L - s)$ ground states. When $L\mu$ is an integer there are additional degeneracies: e.g. for $\mu = 0$, there are ground states that have height 0 on one sublattice and an arbitrary distribution of heights 1 and $L - 1$ on the other sublattice.

Since bulk ground states are usually independent of the anisotropy parameter u, one expects the same bulk ground states for the first ordered limit (u fixed). A direct calculation for a special case in appendix A confirms this; we have no general proof.

2.4. The order parameters

The local height probability (LHP) is given by

$$P(a) = \frac{1}{Z} \sum_{\text{conf.}} \delta(\sigma_1, a) \prod_{\text{faces}} W \text{ (face configuration)}. \tag{2.19}$$

† The usual definition of ground states as configurations with maximum Boltzmann weight is inconvenient here, because those configurations might depend on p.

This is the probability that the height σ_1 at an arbitrary given site, labelled by 1, is a. Before presenting the main steps of the calculation given in [2] we give a simple discussion of the main properties of this function. First of all, as a function of μ it is $2(L-s)$-valued; suppose that μ is not a multiple of $1/L$, then, as $p \to 1$, the system freezes into one of its $2(L-s)$ ground states. It is therefore natural to define $2(L-s)$ functions $P^{(b,c)}(a)$, denoting the probability that the height at site 1 is a, given that the system is in the (b, c) phase and will thus freeze into the ground state chequerboard configuration (b, c) as $p \to 1$. Since this ground state distinguishes between the even and the odd sublattice, there are *two* functions $P^{(b,c)}(a)$, one for each sublattice that site 1 can be part of: these functions are denoted by $P_0^{(b,c)}(a)$ and $P_1^{(b,c)}(a)$ for the even and odd sublattice respectively. They are functions of μ, u and p and they are obviously defined only for values of μ such that (b, c) or (c, b) is a ground state. Explicitly:

$$P_\eta^{(b,b\pm1)}(a) \text{ is defined only when } \left[\frac{sb}{L}+\mu\right]=\left[\frac{s(b\pm1)}{L}+\mu\right] \qquad \eta=0,1. \tag{2.20}$$

Because of the periodicity of the weights, all LHPs are periodic in μ with period 1. The behaviour in the limit $p \to 1$ follows directly from the fround states:

$$P_0^{(b,c)}(a;\mu,u,p=1)=\delta_{ab} \tag{2.21a}$$

$$P_1^{(b,c)}(a;\mu,u,p=1)=\delta_{ac}. \tag{2.21b}$$

When μ is a multiple of $1/L$ there are mixed phases: e.g. for $\mu=0$ there are phases which can be denoted by $(0, \frac{1}{L-1})$ and $(\frac{1}{L-1}, p)$. The LHP for these mixed phases can be calculated as the limit of the LHP for the appropriate pure phase; e.g.

$$P^{(0,\frac{1}{L-1})}(a;0,u,p)=\lim_{u\downarrow 0} P_\eta^{(0,1)}(a;\mu,u,p)=\lim_{u\uparrow 0} P_\eta^{(0,L-1)}(a;\mu,u,p). \tag{2.22}$$

The functions $P_\eta^{(0,L-1)}(a)$ and $P_\eta^{(0,1)}(a)$ have the same limit for $\mu\uparrow 0$ and $\mu\downarrow 0$ respectively; in fact they are the analytical continuation of each other near $\mu=0$. This implies that a system in the $(0, 1)$ phase undergoes a *smooth* transition into the $(0, L-1)$ phase as μ goes through zero from above.

The LHPs $P_\eta^{(b,c)}(a)$ are calculated with the aid of the corner transfer matrix (CTM) method. An explanation and justification of this 'trick' is given by Baxter in his book [8, chapter 13]. There are CTMs for each phase (b, c): the normalized CTM for the lower-right quadrant and phase (b,c) is denoted by $A_{n\eta}^{(b,c)}$ and has elements

$$A_{n\eta}^{(b,c)}{}_{\sigma,\sigma'}=\frac{1}{Z_\eta^{(b,c)}}\,\delta_{\sigma_1,\sigma_1'}\sum_{\text{conf. faces}}\prod W\,(\text{face configuration}) \tag{2.23}$$

where the product is over all faces of the quadrant and the sum over all configurations of the heights on the internal sites; σ and σ' are the height configurations on the left and upper edge respectively. $Z_\eta^{(b,c)}$ is a normalization factor needed to make the largest element of $A_{n,\eta}^{(b,c)}$ equal to unity, i.e. the element corresponding to the configurations

$$\sigma'=\sigma\equiv(\sigma_1,\sigma_2,\sigma_3,\sigma_4,\sigma_5,\ldots)=\begin{cases}(b,c,b,c,b,\ldots) & \text{if } \eta=0\\ (c,b,c,b,c,\ldots) & \text{if } \eta=1.\end{cases} \tag{2.24}$$

CTMs for the upper-right, upper-left and lower-left quadrant are defined similarly, and are denoted by B, C and D respectively. From the symmetries of the weights and

formula (2.7) it follows that

$$C^{(b,c)}_{n\eta}(\mu, u, p) = A^{(b,c)}_{n\eta}(\mu, u, p) \tag{2.25a}$$

$$D^{(b,c)}_{n\eta}(\mu, u, p) = B^{(b,c)}_{n\eta}(\mu, u, p) \tag{2.25b}$$

$$B^{(b,c)}_{n\eta}(\mu, u, p) = A^{(b,c)}_{n\eta}(\mu, \lambda - u, p)G \tag{2.25c}$$

where G has matrix elements

$$G_{\sigma,\sigma'} = [\vartheta_4(w_{\sigma_1}, p)]^{1/2}\delta_{\sigma\sigma'}. \tag{2.26}$$

Defining further the matrix $S(a)$ by

$$S(a)_{\sigma,\sigma'} = \delta_{\sigma_1 a}\delta_{\sigma\sigma'} \tag{2.27}$$

the LHPs for an infinitely large lattice can be expressed as

$$P^{(b,c)}_{\eta}(a) = \lim_{\text{lattice size}\to\infty} \frac{\operatorname{Tr} S(a)A^{(b,c)}_{n\eta}B^{(b,c)}_{n\eta}C^{(b,c)}_{n\eta}D^{(b,c)}_{n\eta}}{\operatorname{Tr} A^{(b,c)}_{n\eta}B^{(b,c)}_{n\eta}C^{(b,c)}_{n\eta}D^{(b,c)}_{n\eta}}. \tag{2.28}$$

In the large lattice limit the normalized CTMs commute for different values of u, so they have a common set of eigenvectors independent of u. Furthermore Baxter shows that rth largest eigenvalue of $A_{n\eta}(u)$ becomes

$$A_r(u) = m_r \exp(-\zeta_r u). \tag{2.29}$$

Here the ζ_r and m_r are independent of u. These numbers can be determined as follows: transform the spectral parameter $u \to u - in \log p$ (n integer); from the property of the theta functions

$$\vartheta_j(z - i\log p, p) = -p^{-1}\,e^{-2iz}\vartheta_j(z, p) \qquad \text{for } j = 1, 4 \tag{2.30}$$

it follows that the face weights α_a, β_a, γ_a and δ_a get multiplied by the following factors respectively:

$$(-1)^n p^{-n^2} e^{-2inu} \cdot \begin{cases} e^{2in\lambda} \\ 1 \\ e^{-2in(w_0+a\lambda)} \\ e^{2in(w_0+a\lambda)}. \end{cases} \tag{2.31}$$

The factor before the brace is the same for all face weights and therefore cancels out of the normalized CTM; when n is a multiple of L, $\exp(2in\lambda) = 1$, so, as far as the normalized CTM is concerned, the transformation $u \to u - ikL\log p$ (k integer) is equivalent to the transformation

$$\alpha_a \to \alpha_a \tag{2.32a}$$

$$\beta_a \to \beta_a \tag{2.32b}$$

$$\gamma_a \to e^{i\varphi}\gamma_a \tag{2.32c}$$

$$\delta_a \to e^{-i\varphi}\delta_a \tag{2.32d}$$

where $\varphi = -2kLw_0$. This gauge transformation can be rewritten more compactly as

$$W\begin{pmatrix} d & c \\ a & b \end{pmatrix} \to W\begin{pmatrix} d & c \\ a & b \end{pmatrix} e^{i\varphi(a+c-b-d)/2} \tag{2.33}$$

and it results in the following transformation on the eigenvalues of $A_{n\eta}$:

$$A_r(u) \to A_r(u - ikL\log p) = \exp(-i\varphi\sigma_{1r}/2 + \text{boundary terms})A_r(u) \tag{2.34}$$

where σ_{1r} is the value of the first height in the eigenvector of A_n corresponding to the rth eigenvalue and the boundary terms arise from gauge factors for the outer boundary of the system. Since these boundary terms depend on the shape of the outer boundary and LHPs for a phase should not depend on the shape of the boundary, the presence of these terms should not affect any of the results and therefore they need not be considered.

It now follows from equations (2.29) and (2.34) that

$$\zeta_r = \frac{2\pi}{kL\varepsilon}(\bar{n}_r - kL\mu\sigma_{1r}/2) \tag{2.35}$$

where ε is defined in equation (2.13) and $\bar{n}_r$ is an integer.

The numbers $n_r = \bar{n}_r - kL\mu\sigma_{1r}/2$ are independent of p and u and therefore they can be determined by considering the ordered limit $p \to 1$, w fixed: it follows from the conjugate nome expressions (2.15a–d) that in this limit the face weights are given by

$$W\begin{pmatrix} d & c \\ a & b \end{pmatrix} = w^{H(d,a,b)/L+(s-L)/2L}\delta_{ac} \tag{2.36}$$

where $H(d, a, b) \in [0, L)$ is given by

$$\begin{aligned} &H(d, d+1, d+2) = H(d, d-1, d-2) = L - s \\ &H(d, d \pm 1, d) = \{\pm(sd + L\mu)\} \bmod L. \end{aligned} \tag{2.37}$$

Note that $H(d, d \pm 1, d)$ has a jump discontinuity as a function of μ when $\pm(sd + L\mu) = 0 \bmod L$. We see from (2.36) that in this limit the lower right CTM for a lattice with boundary as in figure 4 and size m is diagonal and has elements

$$\begin{aligned} A(u)_{\sigma,\sigma'} &= \delta_{\sigma\sigma'} \prod_{j=1}^{m} W\begin{pmatrix} \sigma_j & \sigma_{j+1} \\ \sigma_{j+1} & \sigma_{j+2} \end{pmatrix}^j \\ &= w^{m(m+1)(s-L)/4L + \sum_{j=1}^{m} jH(\sigma_j, \sigma_{j+1}, \sigma_{j+2})/L}. \end{aligned} \tag{2.38}$$

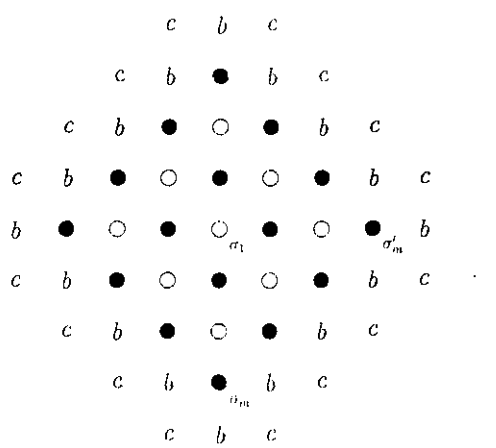

Figure 4. A lattice with the shape suitable for the CTM trick. It has linear size $m = 4$ and the boundary heights are fixed at the values b and c. The even and odd sublattice are denoted by open and closed circles respectively.

The boundary heights σ_{m+1} and σ_{m+2} depend on the phase (b, c) that one is interested in: $(\sigma_{m+1}, \sigma_{m+2}) = (b, c)$. It thus follows that $m_r = 1$ and that

$$n_r = n_\sigma = \tfrac{1}{2}m(m+1)(s-L) + 2\sum_{j=1}^{m} jH(\sigma_j, \sigma_{j+1}, \sigma_{j+2}). \tag{2.39}$$

From equation (2.28) it follows that the LHPs are given by

$$P_\eta^{(b,c)}(a) = \lim_{m\to\infty} \frac{\vartheta_4(w_a) X_{2m+\eta}^{abc}(x^{4s/L})}{\sum_{d=0}^{L-1} \vartheta_4(w_d) X_{2m+\eta}^{dbc}(x^{4s/L})}. \tag{2.40}$$

Here the configuration sums $X_m^{abc}(q)$ are defined as

$$X_m^{abc}(q) = \sum_{\{\sigma_1,\ldots,\sigma_m\}} q^{\sum_{j=1}^{m} jH(\sigma_j,\sigma_{j+1},\sigma_{j+2})} \tag{2.41}$$

with $\sigma_{m+1} = b$, $\sigma_{m+2} = c$. Using the definition (2.41) it is easily seen that configuration sums satisfy the following recursion relation:

$$X_m^{abc}(q) = \sum_d q^{mH(d,b,c)} X_{m-1}^{adb}(q) \tag{2.42}$$

with the initial condition

$$X_0^{abc}(q) = \delta_{ab}. \tag{2.43}$$

The solution of this recursion and the subsequent evaluation of $P_\eta^{(b,c)}(a)$ in terms of theta functions is carried out in [2] and need not be reviewed here. We will just give the results: For L odd, it is found that

$$P_\eta^{(b,c)}(a) = \frac{\vartheta_4(w_a, p)\vartheta_4[a\pi/2L + (c-a)\pi/2 - (n_{bc}\pi + w_0)/[2(L-s)], (-1)^\eta p^{1/4s(L-s)}]}{L\vartheta_4(0, p^{L/s})\vartheta_4[(sn_{bc}\pi + Lw_0)/(L-s), p^{L/(L-s)}]} \tag{2.44}$$

and when L is even

$$P_\eta^{(b,c)}(a) = \begin{cases} \dfrac{2\vartheta_4(w_a, p)\vartheta_4[(a\pi/L) - (n_{bc}\pi + w_0)/(L-s), -p^{1/s(L-s)}]}{L\vartheta_4(0, p^{L/s})\vartheta_4[(sn_{bc}\pi + Lw_0)/(L-s), p^{L/(L-s)}]} & \text{when } a - b = \eta \bmod 2 \\ 0 & \text{otherwise.} \end{cases} \tag{2.45}$$

Here η_{bc} is a function of μ with jump discontinuities:

$$\eta_{bc} = \begin{cases} c - [(sc/L) + \mu] & b - c = 1 \bmod L \\ c - 1 - [(sc/L) + \mu] & b - c = -1 \bmod L. \end{cases} \tag{2.46}$$

It follows that $P_\eta^{(b,c)}(a)$ and also its Fourier transforms

$$S_j^{(b,c)} = \tfrac{1}{2}\sum_{a=0}^{L-1} e^{2\pi i aj/L}\{P_0^{(b,c)}(a) + P_1^{(b,c)}(a)\} \tag{2.47a}$$

$$R_j^{(b,c)} = \tfrac{1}{2}\sum_{a=0}^{L-1} e^{2\pi i aj/L}\{P_0^{(b,c)}(a) - P_1^{(b,c)}(a)\} \tag{2.47b}$$

which are a measure for the height ordering and the sublattice ordering respectively, have jump discontinuities. These jump discontinuities are caused by the discontinuities of $H(c, b, c)$ and lie precisely at one of the borders of the range of validity given by equation (2.20). Therefore we propose that these jumps have no physical meaning. This claim is supported by the observations that:

(i) within their range of validity the LHPs have the expected properties

$$P_{\eta}^{(b,c)}(a) = P_{1-\eta}^{(c,b)}(a) \tag{2.48}$$

and

$$\lim_{p\to 1} P_{\eta}^{(b,c)}(a) = \begin{cases} \delta_{ab} & \text{if } \eta = 0 \\ \delta_{ac} & \text{if } \eta = 1 \end{cases} \tag{2.49}$$

(ii) the LHPs for different phases are the analytical continuation of each other around the regime boundaries: e.g. $P_{\eta}^{(L-1,0)}(a)$ is the analytical continuation of $P_{\eta}^{(1,0)}(a)$ as μ goes through zero from above,

while outside of their range of validity they do not have the properties (2.48) and (2.49).

On the other hand: the $P_{\eta}^{(b,c)}(a)$ as given by equations (2.40) and (2.41) are well defined mathematically and they have an interesting property: when μ is such that (b, c) or (c, b) is not a ground state it is possible to find a ground state (b', c') such that $n_{b'c'} = n_{bc}$ and therefore

$$P_{\eta}^{(b,c)}(a; \mu) = P_{\eta}^{(b',c')}(a; \mu). \tag{2.50}$$

So outside of its range of validity $P_{\eta}^{(b,c)}(a)$ can be interpreted as the LHP for a different phase (b', c'). This interpretation is not very satisfactory, because in equation (2.41) b and c can be interpreted as the heights on the boundary as in figure 4, so the most 'natural' interpretation of $P_{\eta}^{(b,c)}(a)$ would be that it is the LHP for a lattice with boundary heights on the two sublattices fixed at the values b and c, in the limit $m \to \infty$. Whether this interpretation is correct will be examined in the following section.

3. Fixed boundary conditions

3.1. The CSOS model

In this section we consider the CSOS models on a lattice as in figure 4. We would like to calculate the probability that the site in the centre of such a lattice has height a when the boundary heights are fixed at the values b and c. For a finite lattice of linear size m we will denote this probability by $Q_{(m)}^{(b,c)}(a)$, the infinite lattice limit will result in two LHPs $Q_{\eta}^{(b,c)}(a)$, depending on whether m goes to infinity through the even ($\eta = 0$) or odd ($\eta = 1$) integers.

When μ is such that (b, c) is a ground state the LHP $P_{\eta}^{(b,c)}$ has a clear physical interpretation (see equation (2.20)) and there is no doubt that $Q_{\eta}^{(b,c)}$ has a clear physical interpretation (see equation (2.20)) and there is no doubt that $Q_{\eta}^{(b,c)} = P_{\eta}^{(b,c)}$. For non-ground state boundary conditions we would like to know whether $Q_{\eta}^{(b,c)}$ and $P_{\eta}^{(b,c)}$ as defined by equation (2.40) are still equal.

Unfortunately, an analytical calculation of $Q_{\eta}^{(b,c)}$ for all values of μ, u and p is beyond our capacities at the moment: we can calculate them only in the ordered limit $p \to 1$; in this limit the LHPs follow directly from the ground states. The calculation is done for the case $L = 3$, $s = 2$ in appendix B. It turns out that $P_{\eta}^{(b,c)}$ and $Q_{\eta}^{(b,c)}$ are *not* the same.

It is not difficult to see why the two LHPs P and Q can be different for non-ground state boundary conditions: equation (2.29) for the eigenvalues of the normalized transfer matrix was derived by Baxter assuming that the boundary of the system was of no importance. For ground state boundary conditions this assumption is correct, but for non-ground state boundary conditions this may not be the case, since there has to be an interface between the bulk of the system and the boundary, as shown in appendix B. This interface can spoil the nice properties of the CTM. In particular, the neglect of the boundary terms in equation (2.34) is no longer correct.

The discontinuous (first order) anisotropy-dependent transition that is found in appendix B for $x \to 0$ will persist for positive x. However, we expect the locus to change as a function of x so that the sharp edges in figure 7 are rounded off. Whether this *boundary induced* transition is accompanied by essential 'droplet' singularities in the free energy and the order parameters we do not know for sure, but they will not occur in the *bulk* free energy and in the order parameters for a system with free boundaries as calculated in [2] and there is circumstantial evidence—from the Ising model—that the boundary free energy has no essential singularity either.

3.2. The Ising model

A similar anisotropy-dependent transition can also be observed in the Ising model with appropriate boundary conditions as in figure 5. When the ferromagnetic couplings in the horizontal and vertical direction are given by J_1 and J_2 respectively and when the lattice has size $N_1 \times N_2$ the ground state is 'all plus' when $N_1 J_2 > N_2 J_1$ and 'all minus' when $N_1 J_2 < N_2 J_1$. This transition will persist at finite temperatures below T_c.

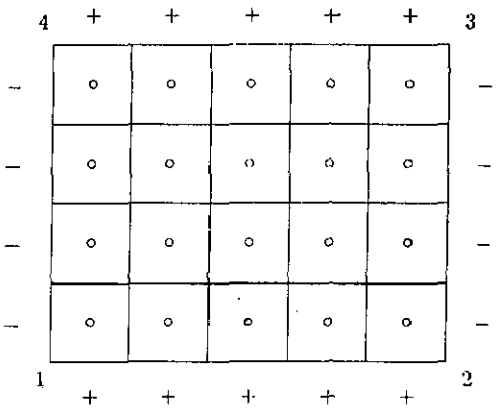

Figure 5. The boundary conditions in the Ising model. The spins reside on the open circles. The dual lattice is also shown.

The partition function of this model can be written in a 'low temperature' expansion as

$$Z = x^{-N_1(N_2+1)/2} y^{-N_2(N_1+1)/2} \sum_{\text{graphs}} x^h y^v \tag{3.1}$$

where $x = \exp(-2K_2)$, $y = \exp(-2K_1)$ and $K_i = \beta J_i$. The sum is over all graphs on the dual lattice consistent with the boundary conditions; h and v are the number of horizontal and vertical bonds of the graph.

Following Vdovichenko [9] we rewrite the sum over graphs as

$$(\det(\mathbf{E}-\mathbf{W}))^{1/2}[\mathbf{M}_{4,1}\mathbf{M}_{3,2}+\mathbf{M}_{2,1}\mathbf{M}_{3,4}-\mathbf{M}_{3,1}\mathbf{M}_{4,2}] \tag{3.2}$$

where $\mathbf{W}$ and $\mathbf{E}$ are matrices of which the indices correspond to directed links of the dual lattice. They can be expressed as

$$\mathbf{E}=1_{N_1+1}\otimes 1_{N_2+1}\otimes 1_4 \tag{3.3}$$

and

$$\begin{aligned}\mathbf{W}= U_{N_1+1}\otimes 1_{N_2+1}\otimes\begin{bmatrix} x & 0 & 0 & 0\\ s\sqrt{xy} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ s^{-1}\sqrt{xy} & 0 & 0 & 0\end{bmatrix}+1_{N_1+1}\otimes U_{N_2+1}\otimes\begin{bmatrix} 0 & s^{-1}\sqrt{xy} & 0 & 0\\ 0 & y & 0 & 0\\ 0 & s\sqrt{xy} & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\\ +\tilde{U}_{N_1+1}\otimes 1_{N_2+1}\otimes\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & s^{-1}\sqrt{xy} & 0\\ 0 & 0 & x & 0\\ 0 & 0 & s\sqrt{xy} & 0\end{bmatrix}\\ +1_{N_1+1}\otimes\tilde{U}_{N_2+1}\otimes\begin{bmatrix} 0 & 0 & 0 & s\sqrt{xy}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & s^{-1}\sqrt{xy}\\ 0 & 0 & 0 & y\end{bmatrix}.\end{aligned} \tag{3.4}$$

Here 1_n denotes the $n\times n$ unit matrix, U_n is the $n\times n$ matrix with elements $(U_n)_{i,j}=\delta_{i,j+1}$ $(1\leqslant j\leqslant n-1)$, $s=e^{i\pi/4}$ and the tilde denotes transposition.

The factor between square brackets in equation (3.2) must be read as follows: the four corners of the dual lattice are numbered counterclockwise starting in the lower left corner as in figure 5. $\mathbf{M}_{i,j}$ is equal to the sum over all 'walks' from corner j to corner i of the quantity

$$(-1)^{\#\text{self-intersections}}x^{\#\text{horizontal steps}}y^{\#\text{vertical steps}}.$$

As implied in [9] it can easily be related to matrix elements of

$$\sum_{l=0}^{\infty}\mathbf{W}^l=(\mathbf{E}-\mathbf{W})^{-1}. \tag{3.5}$$

Explicitly:

$$\begin{aligned}\mathbf{M}_{4,1}=s^{-1}\sqrt{xy}\langle(1,N_2)\uparrow|(\mathbf{E}-\mathbf{W})^{-1}|(1,1)\rightarrow\rangle\\ +s^{-2}x\langle(2,N_2+1)\leftarrow|(\mathbf{E}-\mathbf{W})^{-1}|(1,1)\rightarrow\rangle\\ +y\langle(1,N_2)\uparrow|(\mathbf{E}-\mathbf{W})^{-1}|(1,1)\uparrow\rangle\\ +s^{-1}\sqrt{xy}\langle(2,N_2+1)\leftarrow|(\mathbf{E}-\mathbf{W})^{-1}|(1,1)\uparrow\rangle\end{aligned} \tag{3.6a}$$

$$\begin{aligned}\mathbf{M}_{3,1}=x\langle(N_1,N_2+1)\rightarrow|(\mathbf{E}-\mathbf{W})^{-1}|(1,1)\rightarrow\rangle\\ +s^{-1}\sqrt{xy}\langle(N_1+1,N_2)\uparrow|(\mathbf{E}-\mathbf{W})^{-1}|(1,1)\rightarrow\rangle\\ +s\sqrt{xy}\langle(N_1,N_2+1)\rightarrow|(\mathbf{E}-\mathbf{W})^{-1}|(1,1)\uparrow\rangle\\ +y\langle(N_1+1,N_2)\uparrow|(\mathbf{E}-\mathbf{W})^{-1}|(1,1)\uparrow\rangle\end{aligned} \tag{3.6b}$$

and because of symmetry

$$\mathbf{M}_{3,2} = \mathbf{M}_{4,1} \tag{3.6c}$$

$$\mathbf{M}_{2,1}(x, y, N_1, N_2) = \mathbf{M}_{4,1}(y, x, N_2, N_1) \tag{3.6d}$$

$$\mathbf{M}_{3,4} = \mathbf{M}_{2,1} \tag{3.6e}$$

$$\mathbf{M}_{4,2} = \mathbf{M}_{3,1}. \tag{3.6f}$$

The directed links are labelled by the site from which they originate and the direction in which they point.

The factor between square brackets in equation (3.2) incorporates the effect of the boundary conditions: replacing it by unity gives the partition function for the 'all plus' or 'all minus' boundary condition. So all information about the anisotropy-dependent transition is contained within the last factor.

The contribution to the total free energy from the 'non-groundstate' boundary conditions is therefore

$$\delta F = -\beta \log [\mathbf{M}_{4,1}^2 + \mathbf{M}_{2,1}^2 - \mathbf{M}_{3,1}^2]. \tag{3.7}$$

In the limit of large N_1, N_2 with N_2/N_1 constant δF can develop singularities as a function of x and y. Singularities arise either when one of the $\mathbf{M}_{i,j}$ becomes singular or when one of the $\mathbf{M}_{i,j}$ dominates the other ones in a part of the x-y plane: at the boundaries of these regimes δF gets a cusp.

The first kind of singularity does not occur: the $\mathbf{M}_{i,j}$ themselves are regular. This may be understood as follows: $\mathbf{M}_{i,j}$ is also equal to the two spin correlation function $\langle \sigma_i \sigma_j \rangle$ of two corner spins in the dual model with free boundary conditions and with coupling constants $\tilde{K}_1$ and $\tilde{K}_2$ given by $\tanh \tilde{K}_1 = x$, and $\tanh \tilde{K}_2 = y$. Since for high dual temperatures (i.e. low temperatures in the original Ising model) the two-spin correlation functions decay exponentially, the non-groundstate boundary conditions result in a correction to the free energy per site of order N_1^{-1} which is regular in x and y, apart from a cusp.

For low dual temperatures (i.e. for high temperatures in the original model) the two-spin correlation functions approach a non-zero constant, the square of the corner magnetization, which is regular in x and y, so in this case the correction to the free energy per site is of order N^{-2} and regular.

4. Alternative calculation of the critical indices

The purpose of this section is to show that the critical indices of the CSOS model can be computed by elementary means making use of its relation to the Coulomb gas or Gaussian model. We first note that the CSOS model as defined by [2] can be viewed as a BCSOS model [10] with a field or a modulation of the weights that is periodic in the heights. In this formulation the height variables take all integer values and the weights of the model are invariant for simultaneous translations of the heights by multiples of L. The weights of this periodic BCSOS model are still those defined by equations (2.2(a-d)), with the heights no longer defined mod L, but simply as integers.

As argued in [11] the BCSOS model (without fields) is in its scaling behaviour equivalent to a Gaussian model. Since the Gaussian model can be easily solved, many of the critical properties of the BCSOS model, and therefore also of the CSOS models, follow readily from this equivalence. Without repeating the arguments in full, we

remind the reader that they are based on the assumption of the existence of a renormalization transformation that connects the discrete BCSOS model to a Gaussian model in which the variables are continuous versions of the original heights. All critical exponents of the Gaussian model depend on only one variable. The value of this variable is determined by the knowledge of one exponent, which we have from previous exact solutions, e.g., that of the symmetric eight vertex model [12]. Thus all other exponents follow.

Below we will state in general terms some of the consequences [11] of the equivalence concerning the critical behaviour of the BCSOS models. The starting point is a BCSOS model with weights given by equations (2.11(a-d)). For convenience we choose $u=\lambda/2$, and furthermore we multiply the weight $W(\begin{smallmatrix} d & c \\ a & b \end{smallmatrix})$ by a gauge factor $[\vartheta_4(w_b,p)\vartheta_4(w_d,p)/(\vartheta_4(w_a,p)\vartheta(w_c,p))]^{1/4}$, so that for all values of p the weights are invariant for rotation over $\pi/2$. When the weights are modulated by a factor

$$1+p\cos(\phi+a\omega) \tag{4.1}$$

the free energy of the model changes by an amount proportional to

$$f_s \sim p^{2/y_\omega} \qquad \text{where } y_\omega = 2-\frac{\omega^2}{2\pi(\pi-\lambda)} \tag{4.2}$$

that is only when $y_\omega > 0$. The singular dependence of the free energy on the strength of the field is caused by the collective ordering of the model, so that one value of the heights is more probable than the others. As a result also the Fourier transforms of the LHP acquire a non-zero value, which again depends singularly on the strength of the field:

$$\langle e^{ia\bar{\omega}}\rangle \sim p^{x_{\bar{\omega}}/y_\omega} \qquad \text{where } x_{\bar{\omega}} = 2-y_{\bar{\omega}}. \tag{4.3}$$

The modulation of the weights of the CSOS model slightly away from its critical point is easily computed as the logarithmic derivative of the weights with respect to p at $p=0$. This results in

$$\begin{aligned} \frac{\partial \log \alpha_a}{\partial p} &= \frac{\partial \log \beta_a}{\partial p} = 2(\sin\lambda)^2 \cos[2a(\pi-\lambda)-2w_0] \\ \frac{\partial \log \gamma_a}{\partial p} &= \frac{\partial \log \delta_{a+1}}{\partial p} = 4(\sin\lambda/2)^2 \cos[(2a+1)(\pi-\lambda)-2w_0]. \end{aligned} \tag{4.4}$$

These modulations are written in this form to make apparent the analogy with equation (4.1). This is more clear when one notes that the average height of the configurations with weight α_a and β_a is a and with weight γ_a and δ_{a+1} is $a+\frac{1}{2}$. The frequency of the thermal modulation of the CSOS model is apparently $\omega = 2\pi - 2\lambda$. This immediately leads to the thermal critical exponent $\alpha = 2-2/y_\omega = 2-\pi/\lambda = 2-L/s$, in agreement with [2].

The exponents associated with the Fourier transforms of the LHPs, equations (2.47a, b) can be found similarly. For the symmetric order parameter S_j^{bc} clearly the frequency $\bar{\omega} = 2\pi j/L$. For the antisymmetric order parameter R_j^{bc} one should note that in the BCSOS model the two terms of the summand of equation (2.47(b)) cannot be both non-zero for a model with fixed boundaries, since then each sublattice assumes only even or only odd values. Therefore the two terms can be written in the same form and with the same sign as $\exp[\pi i a(2j-L)/L]P^{(b,c)}(a)$, of which the frequency is

$\bar{\omega} = \pi(2j - L)/L$. The exponents of the symmetric and the antisymmetric order parameter follow from equation (4.3):

$$\bar{\beta}_j = \frac{j^2}{s(L-s)} \qquad \text{and} \qquad \beta_j = \frac{(2j-L)^2}{4s(L-s)}. \tag{4.5}$$

The method presented here is capable only of producing critical exponents. The coefficients of the singular terms cannot be produced, let alone the behaviour further away from the critical point. However, from the above it should be clear that the exponents of a larger class of models can be computed than those for which the analytic solution exists. In the exponents derived above we made no use of the relation between the frequency of the thermal modulation and λ, and therefore we claim for the equations (4.2) and (4.3) a more general applicability.

Acknowledgments

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Appendix A. Bulk ground states

In this appendix we consider bulk ground states for the ordered limit $p \to 1$, u and μ fixed. We use the conjugate nome expressions (2.15(a-d)) with

$$w = x^{2L\nu} \tag{A1}$$

where

$$\nu = \frac{u}{\pi} \in \left[0, \frac{s}{L}\right]. \tag{A2}$$

Leaving out the overall factor $\rho(u)$ the face weights become

$$\alpha_a = x^{\nu(L-s)} \frac{E(x^{2s-2L\nu})}{E(x^{2s})} \tag{A3a}$$

$$\beta_a = x^{s-\nu(L-s)} \frac{E(x^{2L\nu})}{E(x^{2s})} \frac{[E(-x^{2L\mu+2s(a+1)})E(-x^{2L\mu+2s(a-1)})]^{1/2}}{E(-x^{2L\mu+2sa})} \tag{A3b}$$

$$\gamma_a = x^{\nu(s-L+2sa+2L\mu)} \frac{E(-x^{2L(\mu+\nu)+2sa})}{E(-x^{2L\mu+2sa})} \tag{A3c}$$

$$\delta_a = x^{\nu(s+L-2sa-2L\mu)} \frac{E(-x^{2L(\mu-\nu)+2sa})}{E(-x^{2L\mu+2sa})}. \tag{A3d}$$

The weights have the obvious property

$$\omega_a\left(\mu + \frac{s}{L}\right) = \omega_{a+1}(\mu) \qquad \text{for } \omega = \alpha, \beta, \gamma, \delta \tag{A4}$$

and s and L are coprime. Therefore it is sufficient to consider the regime $0 < \mu < 1/L$, the ground states in the other regimes can then easily be deduced.

Table 1. The exponents characterizing the asymptotic behaviour of the face weights as $x \to 0$ for the $L=3$, $s=2$ model in the sector $0<\mu<\frac{1}{3}$: $\omega_a \sim x^{e(\omega,a)}$.

ω	a	$e(\omega, a)$	
α	0	ν	
	1	ν	
	2	ν	
β	0	$3\mu-\nu$	
	1	$1-\nu-3\mu$	
	2	$1-\nu$	
γ	0	$\nu(6\mu-1)$	
	1	$\nu(6\mu+3)$	if $0<\mu<\frac{1}{3}-\nu$
		$\nu(6\mu+3)+2-6(\mu+\nu)$	if $\frac{1}{3}-\nu<\mu<\frac{1}{3}$
	2	$\nu(6\mu+1)$	if $0<\mu<\frac{2}{3}-\nu$
		$\nu(6\mu+1)+4-6(\mu+\nu)$	if $\frac{2}{3}-\nu<\mu<\frac{1}{3}$
δ	0	$\nu(5-6\mu)+6(\mu-\nu)$	if $0<\mu<\nu$
		$\nu(5-6\mu)$	if $\nu<\mu<\frac{1}{3}$
	1	$\nu(1-6\mu)$	
	2	$\nu(3-6\mu)+2+6(\mu-\nu)$	if $0<\mu<\nu-\frac{1}{3}$
		$\nu(3-6\mu)$	if $\nu-\frac{1}{3}<\mu<\frac{1}{3}$

We present the calculation for the model with $L=3$, $s=2$, for simplicity. We first determine the asymptotic behaviour of the face weights as $x \to 0$ using the properties (2.17a-c) of the E-function. It is given by a power law

$$\omega_a \sim x^{e(\omega,a)} \qquad \text{as } x \to 0. \tag{A5}$$

The exponents $e(\omega, a)$ for $0<\mu<\frac{1}{2}$ are given in table 1. From this table one can read off that for $0<\nu<\frac{2}{3}$ the only configurations with non-vanishing weight in the limit $x \to 0$ are the chequerboard configurations (0, 1) and (1, 0): the weight per site $(\gamma_0\delta_1)^{1/2} \to 1$, whereas the weight goes to zero for other configurations.

This shows that, at least for the $L=3$, $s=2$ model, the bulk ground states in the limit $p \to 1$, u fixed are the same as those in the limit $p \to 1$, w fixed.

Appendix B. Ground states for a system with fixed boundaries

In this appendix we calculate the LHPs $Q_\eta^{(b,c)}(a)$ in the limit $p \to 1$, u fixed. Again we can restrict ourselves to the regime $0<\mu<1/L$: it follows from equation (A4) that the LHPs have the property:

$$Q_\eta^{(b,c)}\left(a; \mu+\frac{s}{L}\right) = Q_\eta^{(b+1,c+1)}(a+1; \mu) \tag{B1}$$

and, obviously,

$$Q_\eta^{(b,c)}(a; \mu+1) = Q_\eta^{(b,c)}(a; \mu). \tag{B2}$$

We shall consider the $L=3$, $s=2$ model again. Since for $0<\mu<\frac{1}{3}$ the 'bulk' of the system will be in the (1, 0) phase, there are two ground state candidates, corresponding

to $(\circ, \bullet) = (1, 0)$ resp. $(\circ, \bullet) = (0, 1)$ in the bulk (see figure 4); near the boundary there has to be an interface in order to satisfy both the adjacency requirements and the boundary conditions.

Because of the non-groundstate boundary conditions the weights of the configurations vanish as a power of x in the limit $x \to 0$. The configuration for which this power is lower in the limit $m \to \infty$ is the true ground state for a large lattice with boundary conditions (b, c). This ground state determines the $x \to 0$ limit of the LHP $Q_\eta^{(b,c)}(a)$ in the same way as the bulk ground states determine the $x \to 0$ limit of the LHPs $P_\eta^{(b,c)}(a)$.

In figure 6 the two ground state candidates for the (2, 0) boundary condition and lattice size $m = 4$ are shown. For general values of m both contain $(m-1)(m-2)$ 'bulk' faces of type $\left(\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right)$ and also $(m-1)(m-2)$ faces of type $\left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$; the difference lies in the faces near the boundary.

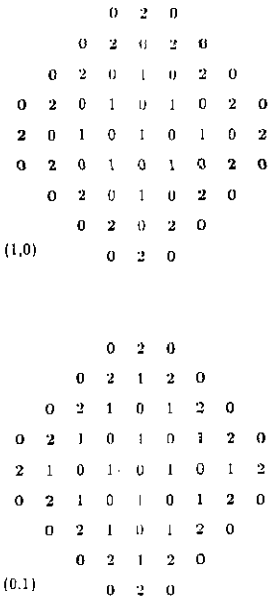

Figure 6. The two ground state candidates for the (2, 0) boundary condition.

The (1, 0) configuration has $2m$ faces of type $\left(\begin{smallmatrix}2 & 0\\ 0 & 2\end{smallmatrix}\right)$ and of type $\left(\begin{smallmatrix}0 & 2\\ 2 & 0\end{smallmatrix}\right)$ and $m-1$ faces of types $\left(\begin{smallmatrix}0 & 2\\ 1 & 0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1 & 0\\ 0 & 2\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0 & 1\\ 2 & 0\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}2 & 0\\ 0 & 1\end{smallmatrix}\right)$, whereas the (0, 1) configuration has m faces of types $\left(\begin{smallmatrix}2 & 0\\ 1 & 2\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1 & 2\\ 2 & 0\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}2 & 1\\ 0 & 2\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0 & 2\\ 2 & 1\end{smallmatrix}\right)$ and $m-1$ configurations of types $\left(\begin{smallmatrix}1 & 2\\ 0 & 1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}0 & 1\\ 1 & 2\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1 & 0\\ 2 & 1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}2 & 1\\ 1 & 0\end{smallmatrix}\right)$. The boundary weights are therefore

$$(\gamma_2\delta_0)^{2m}(\alpha_0\beta_0)^{2(m-1)} \tag{B3}$$

for the (1, 0) configuration and

$$(\alpha_2\beta_2)^{2m}(\alpha_1\beta_1)^{2(m-1)} \tag{B4}$$

for the (0, 1) configuration.

Using the asymptotic behaviour of the face weights from table 1 we readily find the ground states in the limit $m \to \infty$ as a function of μ and ν. The result is shown in figure 7.

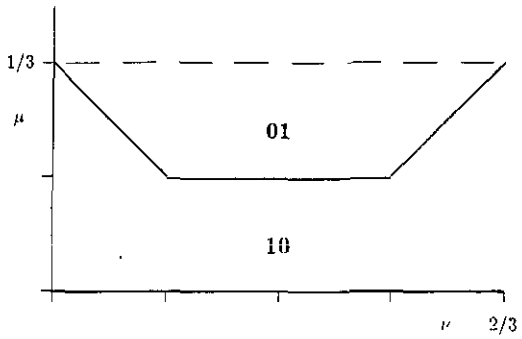

Figure 7. The ground states for the (2, 0) boundary condition in the regime $0<\mu<\frac{1}{3}$ as a function of ν.

For the (2, 1) boundary condition we find a similar picture, whereas the boundary conditions (0, 2) and (1, 2) are like ground state boundary conditions in that they favour the same ground state in the entire regime $0<\mu<\frac{1}{3}$, $0<\nu<\frac{2}{3}$: for (0, 2) boundary conditions the ground state is (0, 1), for (1, 2) boundary conditions the ground state is (1, 0).

The $x\to 0$ limit of $Q_\eta^{(b,c)}(a)$ follows directly from these ground states, so it need not be given separately.

For higher L-values there are more ground state candidates and consequently the determination of the actual ground state becomes more involved, but the overall picture doesn't change: for every boundary condition there is an interval in μ for which the ground state depends discontinuously—in the sense that the ground states on either side of this transition cannot locally be transformed into one another—on the anisotropy ν, except when $s=L-1$ and L is even. In that case there is no discontinuity in the ground state; apparently the interface is not important and therefore $Q_\eta^{(b,c)}(a)=P_\eta^{(b,c)}(a)$. This is of course consistent with the fact that the analytical continuation in μ of $P_\eta^{(b,c)}(a)$ has period 1 in that case.

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